Symplectic Ball Packings: Rigidity, Flexibility, and Beyond

Outline

1. Symplectic: Flexibility v.s. Rigidity

2. Symplectic Ball Packing $\coprod_{i=1}^{N} B(\lambda_i) \xhookrightarrow{s} (M, \omega)$

- $M = B^{2n}(R)$, $N=2$ (Gromov)
- $M = \mathbb{C}P^2$, $\lambda_1 = \cdots = \lambda_N = \lambda$

$\begin{cases} N \le 9 & \text{(McDuff - Polterovich)} \\ N > 9 & \text{(Biran)} \end{cases}$

- Sketch of proof $\begin{cases} \text{Inflation} \\ \text{Taubes SW.} \end{cases}$
- Relation w/ Nagata's Conj.

3. Recent Developments and Open Problems in Symp. Emb.

(1) Stability of ball packing

(2) Gromov width

(3) Ball isotopy

(4) Sympletic embedding of other shapes.

Appendix: Preliminaries in symplectic topology.

1. Symplectic: Flexibility v.s. Rigidity

<u>Def 1</u> $\omega \in \Omega^2(M)$ symplectic if ① $d\omega = 0$

② non-degenerate

<u>Rmk 1</u> ② $\Rightarrow$ $\dim M$ even

· Flexibility

<u>Thm 1</u> (Darboux)

$\forall\, p \in (M^{2n}, \omega)$, $\exists$ local chart $(\mathcal{U}, x_1, y_1, \cdots, x_n, y_n)$ s.t. $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ on $\mathcal{U}$

<u>Rmk 2</u> (i) No local invariants like curvature.

(ii) Thm 1 is proved using Moser's trick.

(iii) More "flexible" results are obtained using h-principle.

e.g. On open mfds, existence of symp. str.

$\Leftrightarrow$ existence of almost complex str.

· Rigidity

<u>Def 2</u> (Symplectic embedding)

$\varphi : (M_1, \omega_1) \xhookrightarrow{C^\infty} (M_2, \omega_2)$ w/ $\varphi^* \omega_2 = \omega_1$, denoted by $(M_1, \omega_1) \xhookrightarrow{S} (M_2, \omega_2)$

<u>Notations</u>: $B^{2n}(a) = \{ z \in \mathbb{C}^n \mid |z|^2 < \frac{a}{\pi} \}$

$Z^{2n}(b) = B^2(b) \times \mathbb{R}^{2n-2}$ $\Big\}$ equipped with $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$

<u>Thm 2</u> (Gromov nonsqueezing) <u>Rmk 3</u> Proved using J-holomorphic curves.

$\exists\, B^{2n}(a) \xhookrightarrow{S} Z^{2n}(b)$ iff $a \le b$.

2. Symplectic Ball Packing

$$\varphi : \coprod_{i=1}^{N} B^{2n}(\lambda_i) \xhookrightarrow{s} (M^{2n}, \omega)$$

Volume constraint : $\sum_{i=1}^{N} \mathrm{Vol}(B^{2n}(\lambda_i)) \leq \mathrm{Vol}(M,\omega) = \frac{1}{n!}\int_M \omega^n \underset{\text{assumption}}{\underline{< \infty}}$

<u>Question 1</u> Does symp. str. impose other obstructions?

<u>Thm3</u> (Gromov Two Balls)

Rmk4 Also proved using J-hol. curve.

$\exists\ B^{2n}(\lambda_1) \sqcup B^{2n}(\lambda_2) \xhookrightarrow{s} B^{2n}(R) \Rightarrow \lambda_1 + \lambda_2 \leq R$.

<u>Def3</u> $\Lambda = \{ \lambda > 0 \mid \exists \coprod_{i=1}^{N} B(\lambda) \xhookrightarrow{s} (M,\omega) \}$

$$v_N(M,\omega) := \sup_{\lambda \in \Lambda} \frac{N \,\mathrm{Vol}\, B(\lambda)}{\mathrm{Vol}(M,\omega)}$$

If $v_N = 1$, say (M,ω) admits a *full packing* (of N equal-sized balls).

<u>Thm4</u> (McDuff-Polterovich '94, Biran '97)

$(M,\omega) = (B^4(1), \omega_0)$

N	1	2	3	4	5	6	7	8	≥ 9
v_N	1	$1/2$	$3/4$	1	$20/25$	$24/25$	$63/64$	$288/289$	1

<u>Rmk5</u> For $N < 9$, $\exists$ packing obstruction as in isometric case (rigid)

$N \geq 9$, $\exists$ full packing as in volume-preserving case (flexible)

- Blow-up & down in symplectic category ($B(\varepsilon) = \{ z \in \mathbb{C}^n \mid |z| \le \varepsilon \}$ in this section)

(1) Blow-down: Remove exc. div. & glue in a ball

$$\begin{tikzcd} & L = \widetilde{\mathbb{C}}^n = \mathcal{O}(-1) \arrow[dl, "\pi"'] \arrow[dr, "pr"] & \\ (\mathbb{C}^n, \omega_0) & & (\mathbb{C}P^{n-1}, \omega_{FS}) \end{tikzcd}$$

$\omega_{FS} = \frac{i}{2} \partial\bar{\partial} f_j$, $f_j(z) = \log\left(\sum_{i=0}^{n-1} |z_i|^2 / |z_j|^2 \right)$ on $\mathcal{U}_j = \{ z_j \neq 0 \} \subseteq \mathbb{C}P^{n-1}$

$\int_{\mathbb{C}P^1} \omega_{FS} = \pi$ Fubini-Study form

Given $Z \overset{C^\infty}{\cong} \mathbb{C}P^{n-1} \hookrightarrow (M^{2n}, \omega)$ w/ $NZ \cong L$ and $\omega|_Z$ symp.

Since $b_2(\mathbb{C}P^n) = 1$, using symp. nbhd thm. & Moser's trick

$\Rightarrow \exists$ nbhd $(\nu Z, \omega) \overset{S}{\cong} (L(\varepsilon), \tilde{\omega}_\lambda)$

where $L(\varepsilon) = \pi^{-1}(B(\varepsilon)) = \{ ([w], z) \in L = \widetilde{\mathbb{C}}^n \mid |z| \le \varepsilon \}$

$\tilde{\omega}_\lambda = \pi^* \omega_0 + \lambda^2 pr^* \omega_{FS} \in \Omega^2(L)$, λ^2 is determined by $\pi \lambda^2 = \int_{\mathbb{C}P^1} \omega$ $\forall$ line $\mathbb{C}P^1 \subseteq Z$.

<u>Lemma 1</u> (7.1.11 in [MS])

$$(L(\varepsilon) \setminus Z, \tilde{\omega}_\lambda) \underset{f}{\overset{S}{\cong}} (B(\sqrt{\lambda^2 + \varepsilon^2}) \setminus B(\varepsilon), \omega_0)$$

<u>Def 4</u> Blow-down of M along $Z := (M - \nu Z) \cup_f B(\sqrt{\lambda^2 + \varepsilon^2})$

It's independent of ε.

(2) Blow-up: Remove a symp. ball & collapse the ∂ via $S^{2n-1} \xrightarrow[\text{Hopf}]{} \mathbb{C}P^{n-1}$

$\psi : B(\lambda) \overset{S}{\hookrightarrow} (M, \omega)$

By Thm 3.1.1 in [MS], we can extend ψ to $B(\sqrt{\lambda^2 + \varepsilon^2}) \overset{S}{\hookrightarrow} (M, \omega)$

Def 5 $\tilde{M}_\psi := (M \setminus \psi(B(\lambda))) \amalg L(\varepsilon) / \sim$, $\tilde{\omega}_\psi := \begin{cases} \omega & \text{on } M \setminus \psi(B(\lambda)) \\ \tilde{\omega}_\lambda & \text{on } L(\varepsilon) \end{cases}$

where $\underset{L(\varepsilon)\setminus Z}{\underset{\cap}{([z], z)}} \sim \underset{M \setminus \psi(B(\lambda))}{\underset{\cap}{\psi\left(\sqrt{1+\frac{\lambda^2}{|z|^2}} \cdot z\right)}}$

$(\tilde{M}_\psi, \tilde{\omega}_\psi)$ is a symplectic blow-up of (M, ω) of weight λ.

It's independent of choice of extension of ψ.

Rmk 6 Blow-down: Topology $\downarrow$, Volume $\uparrow$

Blow-up : Topology $\uparrow$, Volume $\downarrow$

$\Theta : \tilde{M}^{2n} \to M^{2n}$ complex blow-up at p, ω symp. str. on M compatible w/ J

$\begin{aligned} B^{2n}(\lambda) &\xhookrightarrow{\ \iota\ } (M^{2n}, \omega) \\ 0 &\longmapsto p \end{aligned}$. Then the symp blow-up can be performed compatible w/ the cplx blow-up

such that the symp. form $\tilde{\omega}$ on $\tilde{M}$ has $[\tilde{\omega}] = \Theta^*[\omega] - \pi\lambda^2 e$

Here $e = PD(E)$ exc. div.

and $c_1(\tilde{\omega}) = \pi^* c_1(\omega) - (n-1)e$.

See [MS §7.1] for details.

- Sketch of proof for Thm 4 : Settings.

$X_k \to \mathbb{CP}^2$ complex blow-up at k pts. $X_k \overset{C^\infty}{\cong} \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$

$\{L, E_1, \cdots, E_k\}$ canonical basis of $H_2(X_k;\mathbb{Z})$

$\downarrow PD$

$\{l, e_1, \cdots, e_k\}$ basis of $H^2(X_k;\mathbb{Z})$

Write $(d;\vec{m}) = (d; m_1, \cdots, m_k) := dL - \sum_{i=1}^k m_i E_i$

$(\mu;\vec{a}) = (\mu; a_1, \cdots, a_k) := \mu l - \sum_{i=1}^k a_i e_i$

$\|\vec{a}\|^2 = \sum_{i=1}^k a_i^2$, $K = 3L - \sum_{i=1}^k E_i = PD(c_1(X_k))$

$\mathcal{C}_K(X_k) = \{\alpha \in H^2(X_k;\mathbb{R}) \mid \alpha^2 > 0, \exists$ symp. ω w/ $[\omega] = \alpha$ and $c_1(\omega) = PD(K)\}$

$\mathcal{E}_K(X_k) = \{E \in H_2(X_k;\mathbb{Z}) \mid K \cdot E = 1, E \cdot E = -1, \exists S^2 \overset{C^\infty}{\hookrightarrow} X_k$ w/ $[S^2] = E\}$

$\mathcal{E}_\omega(X_k) = \{E \in H_2(X_k;\mathbb{Z}) \mid E \cdot E = -1, \exists$ symp. $S^2 \hookrightarrow X_k$ w/ $[S^2] = E\}$

Def 6 (Cremona transformation)

$\forall k \geq 3$, $Cr : \mathbb{R}^{1+k} \to \mathbb{R}^{1+k}$, $(x_0; x_1, \cdots, x_k) \mapsto (2x_0 - x_1 - x_2 - x_3; x_0 - x_2 - x_3, x_0 - x_1 - x_3,$
$x_0 - x_1 - x_2, x_4, \cdots, x_k)$

$(x_0; x_1, \cdots, x_k)$ is ordered if $x_1 \geq \cdots \geq x_k$

Standard Cremona move : $(x_0;\vec{x}) \mapsto$ ordering of $Cr(x_0;\vec{x})$

An ordered vector $(x_0; x_1, \cdots, x_k)$ is reduced if $x_0 \geq x_1 + x_2 + x_3$ and $x_i \geq 0$ $(\forall i)$

Rmk 7 Cr is the reflection in the (-2)-class $R = L - E_1 - E_2 - E_3$

- Construction & Obstruction of Ball Packing via J-hol. curves

Lemma 2 (Inflation, Lalonde-McDuff)

(M^4,ω) symp. mfd. $A \in H_2(M;\mathbb{Z})$ w/ $A^2 \geq 0$ and $\exists$ closed symp. surface $C \subseteq M$ s.t. $[C] = A$ $\Rightarrow [\omega] + s PD(A)$ has a symp. rep. for $\forall s \geq 0$.

Taubes-Seiberg-Witten Theory

- M closed oriented smooth 4-mfd w/ $b_2^+(M)=1$

$SW : Spin^c(M) \to \mathbb{Z}$ $\quad$ o : orientation of $H^1(M) \oplus H^2_+(M)$

$\mathfrak{s} \mapsto SW(M,o,p,\mathfrak{s})$ $\quad$ p : cnt. component of $\mathcal{P} = \{a \in H^2(M;\mathbb{R}) \mid a^2 > 0\}$

- (Conjugation) $SW(M,o,p,\overline{\Gamma}) = (-1)^{\frac{\chi+\sigma}{4}} SW(M,o,-p,\Gamma)$
- (Wall-crossing) If $c_1(\Gamma)^2 \geq 2\chi(M) + 3\sigma(M)$ and $H^1(M;\mathbb{R}) = 0$

then $SW(M,o,p,\Gamma) - SW(M,o,-p,\Gamma) = 1$

Here o is the orientation of $H^2_+(M;\mathbb{R})$ induced by $-p$.

Thm 5 (Taubes)

(M^4,ω) closed symp. w/ $b_2^+ = 1$ $\quad$ Then $SW(M, o_\omega, p_\omega, \Gamma_\omega) = 1$

$Gr(M,\omega,A)$ "=" $\#\{$J-hol. curves rep. $A \in H_2(M;\mathbb{Z})$ through marked pts$\}/\sim$

Thm 6 (Li-Liu)

(M^4,ω) closed symp w/ $b_2^+(M)=1$. $A \in H_2(M;\mathbb{Z})$ s.t. $A\cdot E \geq -1$, $\forall E \in \mathcal{E}_\omega$

Here $\mathcal{E}_\omega = \{E \in H_2(M;\mathbb{Z}) \mid E\cdot E = -1, \exists$ symp. $S^2 \hookrightarrow M$ w/ $[S^2] = E\}$

Then $SW(M, o_\omega, p_\omega, \Gamma_{\omega, PD(A)}) = Gr(M,\omega,A)$

<u>Cor</u> (1) (Taubes)

X_k = k-fold complex blow-up of $\mathbb{CP}^2$ w/ symp. form ω_k

satisfying $c_1(\omega_k) = PD(3L - E_1 - \cdots - E_k)$ and $[\omega_k](E_j) > 0$ $(\forall j)$

$\Rightarrow$ L is rep. by a symp. emb. S^2.

(2) (Li-Liu)

(M^4, ω) closed symp. $E \in H_2(M;\mathbb{Z})$ satisfies $E \cdot E = -1$ and $c_1(\omega)(E) = 1$

If $\exists$ $S^2 \overset{C^\infty}{\hookrightarrow} M$ rep. E, then $\exists$ symp. $S^2 \hookrightarrow M$ rep. E.

In particular, $\mathcal{E}_K(X_k) = \mathcal{E}_\omega(X_k)$ if $c_1(\omega) = K$.

(3) (Kronheimer-Mrowka, Taubes)

$\alpha \in H^2(X_k;\underline{\mathbb{Q}})$ $\alpha^2 > 0$, $\alpha(E) > 0$ for $\forall E \in \mathcal{E}_K(X_k)$

$\Rightarrow$ $\forall$ symp. form ω on X_k w/ $c_1(\omega) = c_1(X_k)$,

$\exists$ $n \in \mathbb{N}$ s.t. $PD(n\alpha)$ can be rep. by a closed cnt. symp. surface.

Pf. of Cor (1) (2) can be found in [MS, § 13.3]

$\mathcal{C}_{symp}(X_k) := \{ A \in H^2(X_k;\mathbb{Z}) \mid \exists \text{ symp. form on } X_k \text{ w/ } c_1(\omega) = A \}$

<u>Thm</u> (Li-Li)

$\mathrm{Diff}(X_k)$ acts transitively on $\mathcal{C}_{symp}(X_k)$

See [MS, Example 13.4.4]

Thm 8 $\alpha := (\mu; \vec{a}) = \mu l - \sum_{i=1}^{k} a_i l_i \in H^2(X_k; \mathbb{R})$ TFAE:

(1) $\exists \coprod_{i=1}^{k} B^4(a_i) \xrightarrow{S} B^4(\mu)$

(2) $\alpha \in \overline{\mathcal{C}_K}(X_k)$

(3) $\alpha^2 \geq 0$ and $\alpha(E) \geq 0, \forall E \in \mathcal{E}_K(X_k)$

(4) $\|\vec{a}\| \leq \mu$, $a_1, \cdots, a_k \geq 0$, $\sum_{i=1}^{k} a_i m_i \leq \mu d$ for $\forall (d; \vec{m})$ satisfying:

$d, m_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=1}^{k} m_i = 3d - 1$, $\sum_{i=1}^{k} m_i^2 = d^2 + 1$ $(*)$

and reduces to $(0; 0, \cdots, 0, -1)$ under std. Cr. moves.

Rmk 8 (4) $\Leftrightarrow$ (4') $a_1, \cdots, a_k \geq 0$, $\|\vec{a}\| \leq \mu$, $\sum_{i=1}^{k} a_i m_i \leq \mu d$

for $\forall (d, \vec{m})$ satisfying $d, m_i \in \mathbb{Z}_{\geq 0}$, $\sum_{i=1}^{k} (m_i^2 + m_i) \leq d^2 + 3d$

Assuming Thm 8, we can prove Thm 4:

$\forall (d; \vec{m}) \in \mathcal{E}_K(X_k)$, $k(d^2+1) = k \sum_{i=1}^{k} m_i^2 \geq \left(\sum_{i=1}^{k} m_i\right)^2 = (3d-1)^2 = 9d^2 - 6d + 1$

① $k \leq 8 \Rightarrow \# \mathcal{E}_K(X_k) < \infty$

Up to reordering of $\vec{m}$, elements of $\bigcup_{k \leq 8} \mathcal{E}_K(X_k)$ are:

$(0; 0, \cdots, 0, -1)$, $(1; 1, 1)$, $(2; 1^{\times 5})$, $(3; 2, 1^{\times 6})$, $(4; 2^{\times 3}, 1^{\times 5})$

$(5; 2^{\times 6}, 1)$, $(6; 3, 2^{\times 7})$

So one can apply (1) $\Leftrightarrow$ (3) to compute v_k for $k \leq 8$

② $k \geq 9$ Let $(\mu; \vec{a}) = (\sqrt{k}; 1, \cdots, 1)$

$\sum_{i=1}^{k} m_i = 3d - 1 \leq \sqrt{k}\, d$ Apply (4) $\Rightarrow$ (1) we get $\coprod_{i=1}^{k} B^4(1) \xrightarrow{S} B^4(\sqrt{k})$ $\square$

Pf. of Thm 8 (sketch):

(1) $\Leftrightarrow$ (2): It suffices to show: $\coprod_{i=1}^{k} \overline{B}^4(a_i) \xhookrightarrow{s} B^4(\mu)$ iff $(\mu;\vec{a}) \in \mathcal{C}_K(X_k)$

(Nontrivial. See [Schlenk].)

If $\exists \coprod_{i=1}^{k} \overline{B}^4(a_i) \xhookrightarrow{s} B^4(\mu) \xhookrightarrow{s} (\mathbb{CP}^2, \omega_\mu)$

$\overset{\text{blowup}}{\rightsquigarrow} (X_k, \omega_{\mu,\vec{a}})$ w/ $[\omega_{\mu,\vec{a}}] = (\mu;\vec{a})$

$0 < [\omega_{\mu,\vec{a}}]^2 = \mu^2 - \|\vec{a}\|^2$, $c_1(\omega_{\mu,\vec{a}}) = c_1(X_k)$ (See [MS, § 7.1])

$\Rightarrow (\mu;\vec{a}) \in \mathcal{C}_K(X_k)$

Conversely, $\forall$ symp. form ω_k on X_k w/ $[\omega_k] \in \mathcal{C}_K(X_k)$

$\overset{\text{Li-Liu}}{\Rightarrow}$ $\exists$ symp sphere $S_1, \cdots, S_k \subseteq X_k$ rep. $E_1, \cdots, E_k$ $\quad$ disjoint

$\Rightarrow [\omega_k](E_i) > 0 \ (\forall i) \overset{\text{Taubes}}{\Rightarrow} \exists$ symp. sphere $S \in X_k$ rep. L

$\overset{\text{blow down } S_i}{\rightsquigarrow} (\mathbb{CP}^2, S, \omega) \underset{(A4)}{\overset{s}{\simeq}} (\mathbb{CP}^2, \mathbb{CP}^1, \omega_\mu)$

$\Rightarrow \coprod_{i=1}^{k} \overline{B}^4(a_i) \xhookrightarrow{s} (\mathbb{CP}^2 \setminus \mathbb{CP}^1, \omega_\mu) \underset{(A3)}{\overset{s}{\simeq}} B^4(\mu)$

(2) $\Leftrightarrow$ (3):

Symp. Nakai-Moishezon criterion:

$\mathcal{C}_K(X_k) = \{\alpha \in H^2(X_k;\mathbb{R}) \mid \alpha^2 > 0 \text{ and } \alpha(E) > 0 \ \forall E \in \mathcal{E}_K(X_k)\}$ (Δ)

$\subseteq$: $\forall$ symp. form ω w/ $\omega \in \mathcal{C}_K(X_k)$, $\forall E \in \mathcal{E}_K(X_k)$

$\overset{\text{Li-Liu}}{\Rightarrow}$ $\exists$ ω-symp. sphere rep. E.

$\supseteq$: $\alpha = (\mu; \vec{a})$ w/ $\alpha^2 > 0$, $\alpha(E) > 0, \forall E \in \mathcal{E}_K(X_k)$

$\exists\, \varepsilon > 0$ s.t. $\coprod_{i=1}^{k} B^4(\varepsilon a_i) \xhookrightarrow{s} B^4(\mu)$

$\Rightarrow \alpha_\varepsilon = (\mu; \varepsilon\vec{a}) \in \mathcal{C}_K(X_k)$ After rescaling, assume $\mu \in \mathbb{Q}$

Pick $a_i' \in \mathbb{Q}$ s.t. $a_i' > a_i$, $\alpha' = (\mu; \vec{a}')$ satisfies $(\alpha')^2 > 0$, $\alpha'(E) > 0$ $\forall E \in \mathcal{E}_K(X_k)$

$\Rightarrow \exists\, n \in \mathbb{N}$, ω_ε symp. form on X_k s.t.

KMT $[\omega_\varepsilon] = \alpha_\varepsilon$, $PD(n\alpha')$ can be rep. by closed cnt. ω_ε-symp. surface

$\Rightarrow \alpha_\varepsilon + sn\alpha' \in \mathcal{C}_K(X_k)$ (Continuous family of symp. str. have the same c_1)

inflation

$$\Rightarrow \frac{\alpha_\varepsilon + sn\alpha'}{sn+1} = \mu l - \sum_{i=1}^{k} \frac{sna_i' + \varepsilon a_i}{sn+1} e_i \in \mathcal{C}_K(X_k)$$

For s large enough, $\dfrac{sna_i' + \varepsilon a_i}{sn+1} > a_i$

$\Rightarrow \alpha \in \mathcal{C}_K(X_k)$ by (1) $\Leftrightarrow$ (2). So (Δ) holds.

Note that: $\partial \mathcal{C}_K(X_k) = \partial_1 \mathcal{C}_K(X_k) \amalg \partial_2 \mathcal{C}_K(X_k)$

$\partial_1 \mathcal{C}_K(X_k) = \{\alpha \in H^2(X_k;\mathbb{R}) \mid \alpha^2 \geq 0, \alpha(E) \geq 0, \forall E \in \mathcal{E}_K(X_k) \text{ and } \alpha(E) = 0 \text{ for some } E \in \mathcal{E}_K\}$

$\partial_2 \mathcal{C}_K(X_k) = \{\alpha \in H^2(X_k;\mathbb{R}) \mid \alpha^2 = 0, \alpha(E) > 0, \forall E \in \mathcal{E}_K(X_k)\}$

So (2) $\Leftrightarrow$ (3) by (Δ).

(3) $\Leftrightarrow$ (4): $\alpha = (\mu; \vec{a})$, $E = (d; \vec{m})$

$\alpha^2 \geq 0$ iff $\|\vec{a}\| \leq \mu$

$\alpha(E_i) \geq 0$ iff $a_i \geq 0$

$\alpha(E) \geq 0$ iff $\sum_{i=1}^{k} a_i m_i \leq \mu d$

$K \cdot E = 1$ iff $\sum_{i=1}^{k} m_i = 3d - 1$

$E \cdot E = -1$ iff $\sum_{i=1}^{k} m_i^2 = d^2 + 1$

$\forall E \neq E' \in \mathcal{E}_K(X_k)$, $\forall \omega \in \mathcal{C}_K(X_k)$ $\overset{\text{Li-Liu}}{\Longrightarrow}$ E, E' are rep. by ω-symp. S^2

$\Rightarrow \forall E = (d; \vec{m}) \in \mathcal{E}_K(X_k) \setminus \{E_1, \cdots, E_k\}$, $m_i = E \cdot E_i \geq 0$

$\Rightarrow d \geq 1$ since $3d - 1 = \sum_{i=1}^{k} m_i$

To understand the geometric meaning of "reduced to $(0; 0, \cdots, 0, -1)$ by Cr."

consider Dehn twists along (-2)-sphere.

$M^4 \supseteq S \cong S^2$ w/ $S \cdot S = -2$, $\nu S \underset{\text{反定向}}{\cong} D(TS^2) = \{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|x\| = 1, x \perp v, \|v\| \leq 1\}$

Dehn twist along S

$$T_S(x, v) = \begin{cases} \left(\cos(t\pi)x + \sin(t\pi)\frac{v}{\|v\|}, -\|v\|\sin(t\pi)x + \cos(t\pi)v\right) & v \neq 0 \\ (-x, 0) & v = 0 \end{cases}$$

Here $t = 1 - \|v\|$. T_S is given by unit geodesic flow of S^2.

After smoothing, $T_S \in \mathrm{Diff}(M)$. $T_S^2 \underset{C^\infty}{\simeq} \mathrm{id}_M$.

$T_{S*}(B) = B - 2\frac{A \cdot B}{A \cdot A} A = B + (A \cdot B) A$ where $A = [S]$, $B \in H_2(M; \mathbb{Z})$

E.g. $M = X_k$, $A = L - E_1 - E_2 - E_3$ is rep. by a smooth $S^2 \cong S$

$T_{S*} = \mathrm{Cr}$. In fact, A can be rep. by a Lag. sphere S for $\omega \in \mathcal{C}_K(X_k)$ w/ $\omega(A) = 0$

And T_S can be made into a symplectomorphism

called Dehn-Seidel twist.

Cr and transposition $E_i \leftrightarrow E_j$ preserve K and intersection form

and they are induced by diffeomorphisms

$\Rightarrow$ Std. Cr. moves preserve $\mathcal{E}_K(X_k)$

Lemma 3 (Li-Li) X_k $(k \geq 3)$

$\forall (d;\vec{m})$ satisfying Diophantine eq. (Δ)

$(d;\vec{m}) \in \mathcal{E}_K(X_k)$ iff it reduces to $(0;0,\cdots,0,-1)$ by std. Cr. moves.

Pf. $(\Leftarrow)$ is clear. Let's prove $(\Rightarrow)$. Suppose $(d;\vec{m}) \in \mathcal{E}_K(X_k)$

d decreases under std. Cr. moves. Stop when d is min. $\begin{cases} 0 \\ >0 \end{cases}$ and $(d';\vec{m}')$ reduced

If $(d;\vec{m}) \in \mathcal{E}_K$ is reduced, then $E_{d;\vec{m}} \cdot E \geq 0$ $\forall E \in \mathcal{E}_K$ (combinatorial argument)

But $(d;\vec{m})$ has self-intersection -1. Contradiction. $\square$

Rmk 8

(1) Ball packing in high dim (No SW invariant)

Conj $\forall n \geq 3$, $R_1, \cdots, R_k, R \in \mathbb{R}_{>0}$, TFAE:

(1) $\exists \coprod_{i=1}^{k} B^{2n}(R_i) \xhookrightarrow[\text{symp}]{} B^{2n}(R)$

(2) $R_1^n + \cdots + R_k^n < R^n$ (volume obstruction)

and $R_i + R_j < R$ $\forall 1 \leq i < j \leq k$ (two ball obstruction)

(2) Ball packing in other symp. mfd

E.g. (Unobstructed)

Thm 9 (Entov-Verbitsky)

$M = T^{2n}$ w/ Kähler form ω or IHS hyperkähler mfd w/ hyperkähler symp. form ω $\Rightarrow$ $v_N(M,\omega;B^{2n}(1)) = 1$ $\forall N \geq 1$)

· Relation with Nagata's Conjecture

Conj (Nagata)

$\forall k \geq 9$, $\exists$ k pts $p_1, \cdots, p_k \in \mathbb{CP}^2$ s.t. $\forall$ irreducible alg. curve $C \subseteq \mathbb{CP}^2$ passing through each pt. p_i w/ multiplicity $m_i \in \mathbb{N}$, one has

$\Leftrightarrow$ "very general" k pts

$$\deg(C) \geq \frac{1}{\sqrt{k}} \sum_{i=1}^{k} m_i$$

· Nagata proved it when k is a square. Other cases remain open.

· For $k = 2, 3, 5, 6, 7, 8$, the Conj. is wrong. (Nagata)

Thm 10 (McDuff - Polterovich)

Nagata's conj. is true for some k.

$\Rightarrow \exists$ a full symp. packing of B^4 by k equal-sized balls.

Pf. X_k = blow-up of $\mathbb{CP}^2$ at $p_1, \cdots, p_k$

Take $\mu, \alpha \in \mathbb{N}$ s.t. $\frac{1}{\sqrt{k}} > \frac{\mu}{\alpha}$ and $\frac{\mu}{\alpha}$ is arbitrarily close to $\frac{1}{\sqrt{k}}$

$$\rho := \alpha l - \sum_{i=1}^{k} \mu e_i \in H^2(X_k; \mathbb{Z})$$

① $\rho \cdot \rho = \alpha^2 - k\mu^2 > 0$

② $\forall$ irr. curve $\bar{C} \subseteq X_k$ w/ $[\bar{C}] = dL - \sum_{i=1}^{k} m_i E_i$

$$\Rightarrow \langle \rho, [\bar{C}] \rangle = \alpha d - \mu \sum_{i=1}^{k} m_i = \mu d \Big(\underbrace{\frac{\alpha}{\mu}}_{> \sqrt{k}} - \underbrace{\frac{\sum_{i=1}^{k} m_i}{d}}_{\leq \sqrt{k}} \Big) > 0$$

So by Demailly-Paun's Thm,

ρ is a Kähler class. $\Rightarrow \rho \in \mathcal{C}_K(X_k) \Rightarrow l - \frac{1}{\sqrt{k}} \sum_{i=1}^{k} e_i \in \overline{\mathcal{C}_K(X_k)}$

$\Rightarrow \exists$ full packing k equal-sized balls. $\square$

Another proof of Thm 8 (1) ⇒ (4) using ECH capacity (Hutchings)

(X^4, ω) Liouville domain: ω exact and $\exists$ contact form λ on ∂X w/ $d\lambda = \omega|_{\partial X}$

$\lambda \wedge d\lambda > 0$

$\rightsquigarrow$ $0 = c_0(X,\omega) < c_1(X,\omega) \le \cdots \le \infty$ ECH capacities.

ECH

- If $(X_0, \omega_0) \xrightarrow{\text{symp}} (\mathrm{Int}(X_1); \omega_1) \Rightarrow c_k(X_0,\omega_0) \le c_k(X_1,\omega_1)$ $(\forall k)$
  and "$<$" when $c_k(X_0,\omega_0) < \infty$
- $c_k\left(\coprod_{i=1}^n (X_i; \omega_i)\right) = \max\left\{ \sum_{i=1}^n c_{k_i}(X_i;\omega_i) \,\middle|\, \sum_{i=1}^n k_i = k \right\}$
- $E(a,b) = \left\{ (z_1,z_2) \in \mathbb{C}^2 \,\middle|\, \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \le 1 \right\}$, $B(a) = E(a,a)$

  $(a,b)_k :=$ k^{th} smallest entry in the matrix $(am+bn)_{m,n\in\mathbb{N}}$

  $c_k(E(a,b)) = (a,b)_{k+1}$ $\quad c_k(B(a)) = da$ where d is the unique integer

  w/ $\frac{d^2+d}{2} \le k \le \frac{d^2+3d}{2}$

<u>Prop</u> If $\exists \coprod_{i=1}^n B(a_i) \xrightarrow{\text{symp}} B(\mu)$, then $\sum_{i=1}^n a_i m_i \le \mu d$

for $\forall m_1, \cdots, m_n, d \in \mathbb{Z}_{\ge 0}$ (not all zero) satisfying $\sum_{i=1}^n (m_i^2 + m_i) \le d^2 + 3d$

<u>Pf.</u> Let $k_i = (m_i^2 + m_i)/2$, $k = \sum_{i=1}^n k_i$, $k' = (d^2+3d)/2$

$$\sum_{i=1}^n a_i m_i = \sum_{i=1}^n c_{k_i}(B(a_i)) \le c_k\left(\coprod_{i=1}^n B(a_i)\right) \le c_k(B(\mu)) \le c_{k'}(B(\mu)) = \mu d \quad \square$$

3. Developments & open problems in symplectic embeddings.

(1) Stability of ball packing

(M^{2n}, ω), $\mathrm{Vol}(M,\omega) < \infty$, $D \subseteq \mathbb{R}^{2n}$ bounded domain, $\Lambda_k := \{\lambda > 0 \mid \exists \coprod_k \lambda D \xrightarrow[\text{symp}]{} (M,\omega)\}$

D-packing number $v_k(M, \omega; D) := \sup\limits_{\lambda \in \Lambda_k} \dfrac{k \,\mathrm{Vol}(\lambda D)}{\mathrm{Vol}(M,\omega)}$

<u>Thm</u> (Biran, Buse-Hind-Opshtein)

$\forall$ closed symp 4-mfd (M,ω), $\exists N_0 \in \mathbb{N}$ s.t.

$\forall N \geq N_0$, $v_N(M, \omega; B^4(1)) = 1$ (B-packing stability)

<u>Thm</u> (Buse-Hind)

$[\omega] \in H^2(M;\mathbb{Q})$

$\forall$ balls, ellipsoids, polydiscs, <u>rational</u> closed symp. mfd.

have E-packing stability for every ellipsoid E.

<u>Conj</u>. $\forall$ symp. mfd M^{2n} of finite vol. have D-packing stability for $\forall$ bdd domain $D \subseteq \mathbb{R}^{2n}$.

(2) Gromov width

$w_G(M^{2n}, \omega) := \sup\{\pi a^2 \mid B^{2n}(a) \xrightarrow[\text{symp}]{} (M^{2n}, \omega)\}$

<u>Q</u>: (M,ω) closed symp. s.t. $[\omega] = c_1(\omega)$, is $w_G(M,\omega) \geq 1$?

(True when $\dim M = 4$ $(M \underset{C^\infty}{\cong} S^2 \times S^2, \mathbb{CP}^2 \# k\, \overline{\mathbb{CP}^2}$ $(k \leq 8))$

<u>Q</u>: ? $\exists$ closed M w/ symp. forms $[\omega_1] = [\omega_2]$ but $w_G(M,\omega_1) \neq w_G(M,\omega_2)$

(3) Ball isotopy

<u>Thm</u> (McDuff)

$M \cong_{C^\infty} \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ $(k \geq 0)$ w/ symp. form ω or (B^4, ω_0)

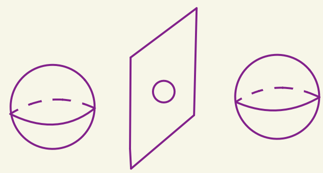
$\Rightarrow \pi_0\left(\mathrm{Emb}_S\left(\coprod_{i=1}^{k} B(\lambda_i), M\right)\right) = \{1\}$.

<u>Q</u>: $n \geq 3$. Is there $0 < \varepsilon < 1$ s.t. $\pi_0\left(\mathrm{Emb}_S\left(B^{2n}(\varepsilon), B^{2n}(1)\right)\right) = \{1\}$?

<u>Q</u>: ? $\exists$ <u>closed</u> (M^{2n}, ω) and $r > 0$ s.t. $\pi_0\left(\mathrm{Emb}_S\left(B^{2n}(r), M\right)\right) \neq \{1\}$

E.g. (Symplectic camel)

$E_S := \mathbb{R}^{2n} \setminus \left\{ z \mid x_1 = 0,\ \sum_j (x_j^2 + y_j^2) \geq \frac{S}{\pi} \right\}$



nonisotopic balls in (E_S, ω_0)

w/ symplectic form ω_0

$\Rightarrow \mathrm{Emb}_S(B^{2n}(r), E_S)$ is disconnected for $r > S$.

$\Rightarrow (E_S, \omega_0) \not\cong_{symp} (\mathbb{R}^{2n}, \omega_0)$ "exotic symp. str."

(4) Symp. emb. of other shapes.

<u>Q</u>: $n \in \mathbb{N}$, $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ w/ $0 < a_1 \leq a_2 \leq \cdots \leq a_n$

$E(a) := \left\{ z = (z_1, \cdots, z_n) \in \mathbb{C}^n \,\middle|\, \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} < 1 \right\}$ w/ symp. form ω_0.

When do we have $E(a) \underset{symp}{\hookrightarrow} E(b)$?

- $n = 1$ $E(a) \underset{symp}{\hookrightarrow} E(b)$ iff $a_1 \leq b_1$
- $n = 2$ Define $0 < c_1(a) \leq c_2(a) \leq \cdots$ by ordering $\{ n_1 a_1 + n_2 a_2 \mid n_i \in \mathbb{Z}_{\geq 0}, n_1 + n_2 > 0 \}$ with multiplicity.

(ECH capacity)

Thm (Hutchings, McDuff)

$\exists\, E(a) \underset{symp}{\hookrightarrow} E(b)$ iff $c_k(a) \le c_k(b)$, $\forall k \in \mathbb{N}$ "Hofer Conj."

- $n \geq 3$ widely open. Analogue of Hofer's Conj. is False. (Guth)

Q: $P(a_1, \cdots, a_n) = D(a_1) \times \cdots \times D(a_n) \subseteq \mathbb{R}^{2n}$ polydisk w/ $a_1 \le \cdots \le a_n$

If $\exists\, P(a_1, \cdots, a_n) \underset{symp}{\hookrightarrow} P(b_1, \cdots, b_n)$, then $a_1 \le b_1$ by nonsqueezing thm.

? $\exists\, b < \infty$ s.t. $P(1, a, a) \underset{symp}{\hookrightarrow} P(b, b, \infty)$ $\forall a \geq 1$ YES (Guth, Hind-Kerman) $> b = 2$

? $\exists\, b < \infty$ s.t. $P(1, \infty, \infty) \underset{symp}{\hookrightarrow} P(b, b, \infty)$ YES (Pelayo-Ngoc)

$\Rightarrow$ No intermediate symp. capacities w/
$$\begin{cases} c(\lambda U) = \lambda^2 c(U) \\ U \overset{s}{\hookrightarrow} V \Rightarrow c(U) \le c(V) \\ 0 < c(B^{2n}(1)) < \infty \\ c(B^{2k}(1) \times \mathbb{C}^{n-k}) = \infty \text{ for some } k \geq 1. \end{cases}$$

Appendix: Preliminaries in Symplectic Topology

Thm A1 (Moser stability)

M closed, smooth. ω_t $(0 \le t \le 1)$ symp. forms on M w/ $[\omega_t] \equiv [\omega_0]$

$\Rightarrow \exists$ C^∞ family of diffeomorphisms ψ_t of M s.t. $\psi_0 = id_M$, $\psi_t^* \omega_t = \omega_0$.

Thm A2 (Symplectic nbhd, Weinstein)

$j = 0,1$ $\quad Q_j \hookrightarrow (M_j, \omega_j)$ cpt. symp. submfds. $\quad \exists$

$$\begin{tikzcd} NQ_0 \arrow[r, "\Phi", "\cong"'] \arrow[d] & NQ_1 \arrow[d] \\ (Q_0, \omega_0) \arrow[r, "\phi", "\overset{s}{\cong}"'] & (Q_1, \omega_1) \end{tikzcd}$$

$\Rightarrow \phi$ extends to a symplectomorphism

$\psi : (\nu Q_0, \omega_0) \overset{s}{\underset{\cong}{\longrightarrow}} (\nu Q_1, \omega_1)$ such $d\psi = \Phi$ on $NQ_0 = (TQ_0)^{\perp\omega}$

Prop A3

$$\begin{array}{ccc} S^{2n+1} & \overset{i}{\hookrightarrow} & (\mathbb{C}^{n+1}, \omega_{n+1}) \\ \pi \downarrow & & \\ (\mathbb{CP}^n, \omega_{FS}) & & \end{array}$$

$\pi^* \omega_{FS} = i^* \omega_0$

$S = \{[0 : u_1 : \cdots : u_n] \in \mathbb{CP}^n\} \overset{C^\infty}{\cong} \mathbb{CP}^{n-1}$

$\Rightarrow \quad f_0 : (B^{2n}(1), \omega_n) \longrightarrow (\mathbb{CP}^n \setminus S, \omega_{FS})$ is a symplectomorphism

$(z_1, \cdots, z_n) \mapsto [\sqrt{1 - |z|^2} : z_1 : \cdots : z_n]$

Thm A4 (Gromov-McDuff)

ω symp. form on $\mathbb{CP}^2$, $S \subseteq \mathbb{CP}^2$ symp. sphere

$\Rightarrow (\mathbb{CP}^2, S, \omega) \overset{s}{\cong} (\mathbb{CP}^2, \mathbb{CP}^1, \omega_\mu = \mu \omega_{FS})$ Here $\mu\pi = \int_S \omega$

Reference

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[Schlenk] Schlenk. Symplectic embedding problems, old and new

[MP] McDuff, Polterovich Symplectic packings and algebraic geometry

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[Hutchings] Quantitative embedded contact homology.